Energy level statistics and localization in sparsed banded random matrix ensemble

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Abstract. The matrix elements of the quantum Hamiltonian, which corresponds to the classical KAM system, $\hat{H} = \hat{H}_0 + \epsilon \hat{V}$ (where $\hat{H}_0$ is integrable) in an energy ordered eigenbasis of $\hat{H}_0$ are considered. Their statistical properties are comprised in the definition of the sparsed banded random matrix ensemble (SBRME). In analysing the spectral statistics of this ensemble we find the power law level repulsion and the level spacing distribution which is well described by the Brody distribution. The numerical study of SBRME shows two interesting scaling laws: (i) the connection between the scaling variable $x = \alpha b^{3/2}$ (α = mean diagonal increment, b = bandwidth) and the level repulsion parameter β as deduced from the Brody distribution, and (ii) the connection between the same scaling variable x and an expression which contains two types of localization length (entropy and geometric). The universal aspects and the importance of SBRME for the Hamiltonian systems in the transition region between integrability and chaos are discussed.

1. Introduction

In this work we investigate a random matrix ensemble which is designed to model the spectral statistical properties such as the energy level distribution of quantum Hamiltonian systems in the transition region between integrability and chaos. Our aim is to understand to what extent this transition region behaves universally. It has been suggested that certain universal aspects can be captured by a one parameter family of nearest-neighbour level spacing distributions $P(S)$ which would uniquely interpolate between the integrable case (Poisson distribution) and the ergodic case (Wigner distribution) (Robnik 1987, Wintgen and Friedrich 1987, Hönig and Wintgen 1989, Prosen and Robnik 1992a). The most frequently used candidate for such a family of distributions is the Brody distribution (Brody 1973, Brody *et al* 1981)

$$P_\beta^B(S) = aS^\beta \exp\left(-bS^{\beta+1}\right) \qquad a = (\beta+1)b \qquad b = \left\{\Gamma\left(\frac{\beta+2}{\beta+1}\right)\right\}^{\beta+1} \qquad (1)$$

which has a simple analytic form but no profound physical justification. It is not yet completely understood why it usually gives a *statistically significant* fit when applied to the finite although very large spectra of up to a few thousand consecutive levels, in particular for small S (see e.g. Prosen and Robnik 1992a), and this is the main motivation of the present work.

On the other hand there are the asymptotic semiclassical Berry–Robnik formulae (Berry and Robnik 1984) for the level spacing distribution $P(S)$ in the transition region. It has been argued by Robnik (1987) that these formulae might not be correct for small values of level spacings S and in a separate work (Prosen and Robnik 1992a) we have found indeed that even for quite large values of S the convergence towards the semiclassical limit is very slow. Berry–Robnik formulae are non-universal in the sense that they depend on the dynamical structure of the classical phase space (the relevant parameters are the volumes of integrable and separate chaotic components) of each particular system and are therefore multiparametric. There is no reason why they should not apply in the semiclassical limit but the convergence towards this limit is usually very slow as is discussed in detail in Prosen and Robnik (1992a, b). The first numerical indication that Berry–Robnik formulae might not be adequate at small S in the *non-semiclassical* limit has been given by Seligman and Verbaarschot (1985) and by Seligman *et al* (1985). The Brody formula and the Berry–Robnik formulae are incompatible, especially for small level spacings where the Brody formula unlike the Berry–Robnik formulae shows the power law level repulsion. We have reasons to expect that the Brody formula (or more precisely, the Izrailev formula, (Izrailev 1989)) should apply approximately for non-semiclassical spectra, especially at small S, whereas the Berry–Robnik formulae should be correct in the semiclassical limit.

The Brody-like distributions such as devised by Izrailev (1989) also shows up in cases of strong localization in classically chaotic time dependent systems. There are two important research programs in this direction: the kicked rotator and an ensemble of banded random matrices with zero increment diagonal (Izrailev 1988, 1989, Casati *et al* 1990, 1991), and second, the modelling of classically ergodic time independent Hamiltonian systems by means of an ensemble of banded random matrices having the diagonal with non-zero increment (BRME) (Wilkinson *et al* 1991, Feingold *et al* 1991), which is based on semiclassical estimates of the matrix elements by Feingold and Peres (1986) and by Wilkinson (1988). This work motivated us to generalize the BRME to also include the systems in the transition region between integrability and chaos. Very closely related results on BRME and on sparsed (but not banded) random matrices have also been recently published by Fyodorov and Mirlin (1991a, b) and by Mirlin and Fyodorov (1991); see also Evangelos and Economou (1992). Fyodorov and Mirlin (1991b) find the correct scaling of the localization length (which was in error in Wilkinson *et al* (1991), as discussed in section 3).

One should be aware of the meaning of substituting a certain Hamiltonian matrix which corresponds to a specific dynamical system by a random member of a given random matrix ensemble. Such random matrix ensembles are usually specified by a small number of parameters (two in the case of BRME and none in the well known cases of GOE, GUE and GSE) and they can therefore capture only the average features of dynamical systems. Only those features which behave universally to a certain extent (do not depend on the details of dynamical systems, e.g. on the fine structure of classical phase space or dynamical correlations etc.) can be reproduced by a suitably chosen parametric ensemble of a few random matrices. In the next section we define a three-parametric ensemble of sparsed banded random matrices (SBRME) which incorporates the statistical properties of the distribution of the matrix elements but suppresses all higher correlations among them. In section 3 we present the numerical results on the level spacing distribution for SBRME matrices which exhibit

the Brody-like behaviour, and suggest two interesting scaling laws for SBRME. The last section is devoted to the discussion and conclusions.

2. Definition of the sparsed banded random matrix ensemble

We start with an integrable Hamiltonian $H_0(\hat{I})$ with f degrees of freedom. By I, θ we denote the canonical action-angle variables and by $\hat{I}, \hat{\theta}$ the corresponding quantum operators with $[\hat{I}_k, \hat{\theta}_l] = (\hbar/i)\delta_{kl}$. We apply the semiclassical torus quantization

$$\hat{I}|m\rangle = I_m|m\rangle \qquad I_m = \hbar(m + \alpha/4) \approx \hbar m \qquad m_k = 0, 1, 2, \ldots \tag{2}$$

where for our purposes we can neglect the Maslov indices α. The semiclassical eigenfunctions $|m\rangle$ and energies E_m read

$$\psi_m(\theta) = \langle\theta|m\rangle = (2\pi)^{-f/2}e^{im\cdot\theta} \tag{3}$$

$$E_m = \langle m|\hat{H}_0|m\rangle. \tag{4}$$

Now we introduce the general perturbed Hamiltonian

$$\hat{H} = \hat{H}_0 + \epsilon V(\hat{I}, \hat{\theta}) \tag{5}$$

which for generic perturbation $\hat{V}$ and small values of the perturbation parameter ϵ represents a quantized KAM system. The general Hermitian perturbation can be expanded as a Fourier series on a torus

$$V(\hat{I}, \hat{\theta}) = \tfrac{1}{2}\sum_m \left\{ K_m(\hat{I})e^{im\cdot\hat{\theta}} + e^{-im\cdot\hat{\theta}}K_m^*(\hat{I}) \right\} \tag{6}$$

with the classical limit

$$V(I, \theta) = \sum_m K_m(I)e^{im\cdot\theta}. \tag{7}$$

The zeroth-order component can be absorbed in the integrable part, so we may put $K_0 := 0$. The realness of the classical limit implies

$$K_m^*(I) = K_{-m}(I). \tag{8}$$

Using the formula

$$e^{ik\cdot\hat{\theta}}|m\rangle = |m + k\rangle \tag{9}$$

which follows from the form of the semiclassical eigenfunctions on a torus (3) one obtains

$$\langle m|\hat{H}|n\rangle = H_0(\hbar m)\delta_{mn} + \tfrac{1}{2}\epsilon \left\{ K_{m-n}(\hbar m) + K_{m-n}(\hbar n) \right\}. \tag{10}$$

Now we shall make certain estimates about the decay of the Fourier components K_m which follow from the Parseval equation

$$\sum_m |K_m(I)|^2 = (2\pi)^{-f} \int d^f \theta \, |V(I, \theta)|^2 = U^2(I). \tag{11}$$

We assume finiteness of the average perturbation on every torus I, i.e.

$$U(I) < \infty \qquad \text{for every } I. \tag{12}$$

The convergence of the sum (11) implies the decay of the Fourier components with increasing index m which must be faster than the power

$$|K_m|^2 < C(I)|m|^{-f} \tag{13}$$

provided that K_m is at least approximately isotropic in m asymptotically (it must not have qualitatively different behaviour in different directions which is a reasonable assumption for a generic perturbation). Since the Fourier components of a generic perturbation decay fast enough it is convenient to assume that the series (7) is finite, i.e. we restrict our class of perturbations to those which have finite Fourier expansions in canonical angles and assume that they are still generic†. So let us assume that there is a maximal magnitude of index M, such that $K_m = 0$ if $|m| > M$ and that all Fourier components are of the order $O(1)$ (all prefactors can be absorbed in the perturbation parameter ϵ). Using equation (10) we have

$$\langle m|\hat{H}|n\rangle = \begin{cases} E_m & m = n \\ O(\epsilon) & |m - n| \leqslant M \\ 0 & |m - n| > M. \end{cases} \tag{14}$$

The next step towards the final form of the Hamiltonian matrix is the ordering of the basis $|m\rangle$ with respect to the energies of the integrable part E_m. Let us introduce Greek integer indices to label the ordered eigenenergies of $\hat{H}_0$, $\lambda < \mu \Rightarrow E_\lambda < E_\mu$, and index vectors such that, $E_{m_\lambda} = E_\lambda$. Now consider two energy surfaces E_λ and E_μ which are close to each other ($\Delta E = E_\mu - E_\lambda \ll E = (E_\mu + E_\lambda)/2$) and the energy shell in between in action space I. The matrix element $H_{\lambda\mu} := \langle m_\lambda|\hat{H}|m_\mu\rangle$ can be non-zero only if the distance between the energy surfaces in action space ΔI is smaller than $M\hbar$ as follows from (14). There are $\Delta I A(E)/\hbar^f = \rho(E)\Delta E$ levels in the energy shell where $A(E)$ is the area of the energy surface in action space which is a smooth classical function of the energy, and $\rho(E)$ is the density of states of the integrable part $\hat{H}_0$ and it is of the order $O(\hbar^{-f})$. We put $\Delta I := \hbar M$ and find that there can be at most $M\hbar A(E)/\hbar^f$ energy levels between E_λ and E_μ to obtain the non-zero matrix element $H_{\lambda\mu}$. It follows that the matrix $H_{\lambda\mu}$ is banded with the bandwidth

$$b = M A(E)\hbar^{1-f}. \tag{15}$$

† We could equivalently perform the whole derivation for the general infinite Fourier expansions and at the end neglect vanishingly small matrix elements within the wings of the dominant band. The restriction to the class of finite Fourier expanded perturbations does not present any serious limitations or loss of generality.

But $\mu - \lambda < b$ is only a necessary condition for the matrix element $H_{\lambda\mu}$ to be non-zero. We must further require (as follows from (14)) that the points $\hbar m_\lambda$ and $\hbar m_\mu$ are actually close together which is typically not the case. It is illuminating to imagine a f-dim ball of radius $\hbar M$ around the point $\hbar m_\lambda$ which by (14) contains all lattice points m_ν for which $H_{\lambda\nu} \neq 0$. The intersection of this ball and the other energy surface E_μ is a $(f-1)$-dim ball with the area of order $a = (\hbar M)^{f-1}$ where the 'gamma function' prefactors and relatively weak dependence on the dimensionless ratio $\Delta I/(\hbar M)$ have been omitted. Since we know that lattice points of increasing energy generally lie randomly in action space (Berry and Tabor 1977) we can interpret the ratio $a/A(E)$ as the *statistical probability* that the matrix element $H_{\lambda\mu}$ will be non-zero. Our matrix $H_{\lambda\mu}$ is therefore sparsed and banded with the density of non-zero elements inside the band or *sparsity* equal to

$$s = \frac{M^{f-1}}{A(E)} \hbar^{f-1}. \tag{16}$$

Since the non-zero elements are distributed rarely and randomly inside the band we are tempted to make the statistical assumption which transforms the Hamiltonian matrix $H_{\lambda\mu}$ into the realization of the statistical ensemble of random matrices. We assume that magnitudes of non-zero elements are Gaussian† random variables with zero mean and the second moment equal to ϵ.

It is suitable to divide the matrix $H_{\lambda\mu}$ by ϵ to minimize the number of parameters and define the sparsed random matrix ensemble (SBRME). An infinite-dimensional (or N-dimensional where N is large enough) matrix A_{jk} is a member of SBRME(α, b, m) if the diagonal elements are independent ordered 'Poissonian events' with the average increment α, $\langle A_{j+1,j+1} - A_{jj} \rangle = \alpha$, the offdiagonal elements are exactly zero outside the band, $A_{jk} = 0$ if $|j - k| > b$, but those inside the band are non-zero with probability m/b, and the non-zero offdiagonal elements are mutually independent Gaussian variables with zero mean and the second moment unity. m is thus the average number of non-zero elements in each row on the one (say right) side of the diagonal.

For the connection between SBRME and real physics it is essential to know the scalings of the ensemble parameters α, b, m with the perturbation parameter ϵ and $\hbar$ which can be deduced from (15) and (16)

$$\alpha = \frac{\rho}{\epsilon} = O\left(\hbar^f/\epsilon\right) \qquad b = O\left(\hbar^{1-f}\right) \qquad m = O(1). \tag{17}$$

SBRME is a kind of generalization of the BRME introduced by Wilkinson *et al* (1991); the latter very closely resembles the special case of SBRME for $m = b$. (The only non-essential difference between BRME(α, b) and SBRME($\alpha, b, m = b$) is that in the former case the diagonal is also Gaussian random whilst in the latter case the diagonal is Poissonian random.) However, we are interested in the other (semiclassical) limit of KAM systems when the sparsity $m/b \rightarrow 0$ since $b \rightarrow \infty$ as $\hbar \rightarrow 0$ and m is fixed and depends only on the 'geometry'. Note that sparsity s goes to zero in the semiclassical limit $\hbar \rightarrow 0$ for a fixed value of ϵ.

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† This choice seems the most obvious one, although we believe that any other smooth distribution would give the same results since it would only alter the class of admissible perturbations which should again show generic features.

3. Numerical results

We study numerically the energy level statistics and the localization properties of the eigenvectors. So far we have not been able to derive any of our results analytically. But we have tried to generalize the ideas of Wilkinson *et al* (1991) for the derivation of the scaling variable and the scaling property of the localization length. However, this attempt was unsuccessful due to the complications connected with the sparsity. When we numerically diagonalize SBRME matrices with small sparsity $m/b \ll 1$ we find that the eigenvectors are localized but not simply exponentially, for they are rather typically 'sparsed'. 'Sparsed' means that only a few components of the eigenvector substantially differ from zero in one localization region. To describe this phenomenon we define two types of localization length of a normalized eigenvector x_j, $\sum_j |x_j|^2 = 1$. The commonly used *entropy localization length* is defined in terms of information entropy (Izrailev 1988, 1989)

$$L_e = 2.075 \exp \left(- \sum_j |x_j|^2 \ln |x_j|^2 \right) \tag{18}$$

and measures *the number of components which substantially differ from zero*. In addition to the entropy localization length we define the *geometric localization length* as

$$L_g = \sqrt{ \sum_j j^2 |x_j|^2 - \left\{ \sum_j j |x_j|^2 \right\}^2 }. \tag{19}$$

In the case of exponentially localized eigenvector $x_j \propto \exp(-|j|/r)$ we have $L_e \approx 8L_g$. But for the sparsed eigenvector we have typically $L_e \ll 8L_g$.

Since we have limited computer capabilities we are not able to explore the whole three-dimensional parameter space (α, b, m) of the SBRME, and therefore we restrict our research to the fixed value of m, $m = 1$. Parameter m is the only ensemble parameter which does not depend on $\hbar$ or on perturbation parameter ϵ, and that is the reason why we have chosen the 'plane of fixed m'. Our SBRME matrices then have on average only three non-zero elements (including the diagonal one) in each row. We have chosen the square mesh of parameter values for $\alpha = 0.0125$, 0.0178, 0.025, 0.0354, 0.05, 0.0707, 0.1, 0.1414, 0.2 and $b = 10, 14, 21, 32, 48, 72$. NAG FORTRAN double precision routines have been used to calculate all eigenvalues for 80 realizations of SBRME of the dimension $N = 2000$ for each pair of values (α, b). The upper and lower thirds of the spectra were discarded due to possible finite size effects, so that only the middle 666 levels were unfolded to the unit mean level spacing and fitted by the Brody formula (1). For each fit we have joined all $80 \times 665 = 53\,200$ level spacings together so that we have obtained the level repulsion exponent β with significant accuracy. Then the full diagonalization (eigenvalues and eigenvectors) was made for SBRME matrices of dimension $N = 800$† to determine the average localization lengths L_e and L_g for all pairs of (α, b). Each average localization length was determined as the double average, first over all eigenvectors of a given SBRME matrix (only eigenvectors corresponding to the lower and upper

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† Full diagonalization is a O (N^3) process, whilst the calculation of the spectrum of a symmetric banded matrix can be performed with only O $(N^2 b)$ operations ($b \ll N$).

100 levels were discarded due to the finite size effects) and then over 10 realizations of an ensemble for each pair of values (α, b). We have obtained three important numerical results about the properties of the ensemble SBRME$(\alpha, b, 1)$.

We have found that the Brody formula (1) gives a statistically significant fit for all parameter values (α, b) that we have chosen. The same χ^2 test as in Prosen and Robnik (1992a) was performed and the typical values of χ^2 were mainly two to four times smaller than the total number of level spacings $80N$. This fact demonstrates the amazing robustness of the level spacing distribution and the applicability of the Brody formula.
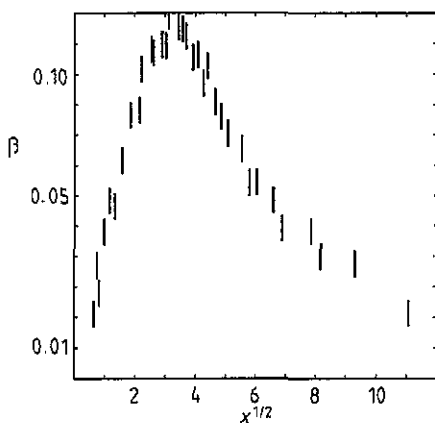

Figure 1. The figure shows the level repulsion exponent β against $\sqrt{x}$, where x is the scaling variable $x = \alpha b^{3/2}$. The fact that all points lie on the one-dimensional curve within the statistical error which is around 0.005 suggests that β should be a unique function of x alone. On the other hand an attempt to plot β against $\sqrt{\alpha b}$ or β against $\sqrt{\alpha b^2}$ etc. leads to a clearly two-dimensional clustering of the experimental points (not shown).

The following function of the parameters

$$x = \alpha b^{3/2} = O\left(\hbar^{(3-f)/2}/\epsilon\right) \tag{20}$$

plays the role of the scaling variable since the level repulsion parameter β is numerically found to be only a function of x, $\beta = \beta(x)$ (see figure 1). It is surprising that the form of the scaling variable is the same as for BRME (Wilkinson *et al* 1991) although the shape of the curve $\beta(x)$ is different. One can verify that $\beta(x)$ gives the correct limiting behaviour as the perturbation parameter ϵ goes to zero, since $\epsilon \to 0 \Rightarrow x \to \infty, \beta \to 0$. In other words, as $\epsilon \to 0$ we indeed observe the approach to the Poissonian level spacing distribution. On the other hand one gets an incorrect limit $\beta(\infty) = 1$ (see Wilkinson *et al* 1991) if one replaces SBRME by BRME (m depends on b, $m = b$ instead of being fixed, $m = 1$) so that the sparsity is surely crucial and governs the behaviour of SBRME. However the behaviour of the semiclassical limit $\hbar \to 0$, which depends on the number of freedoms f, see (20), is not easily understood. We should stress once again that of course we cannot expect the correct semiclassical limit since for dynamical systems it must be given by the Berry–Robnik formulae.

The third numerical result concerns the localization properties of the eigenvectors. One can construct the parameter which controls the level repulsion $\gamma = \alpha L_g/\delta E$. If y^j is an eigenvector localized around index j then δE is an expected difference between jth diagonal element A_{jj} and the corresponding eigenenergy E_j, $\delta E = \langle |A_{jj} - E_j| \rangle$. If δE is large enough ($\delta E \gg \alpha L_g, \gamma \ll 1$) then the two

eigenvectors whose peaks are more than one geometric localization length apart (the corresponding diagonal elements are expected to differ by more than αL_g) can still have similar energies although they cannot 'feel' each other. That means possible level degeneracies i.e. no level repulsion. The opposite case $\gamma \gg 1$ means that two eigenvectors with small energy difference necessarily overlap which leads to the level repulsion.
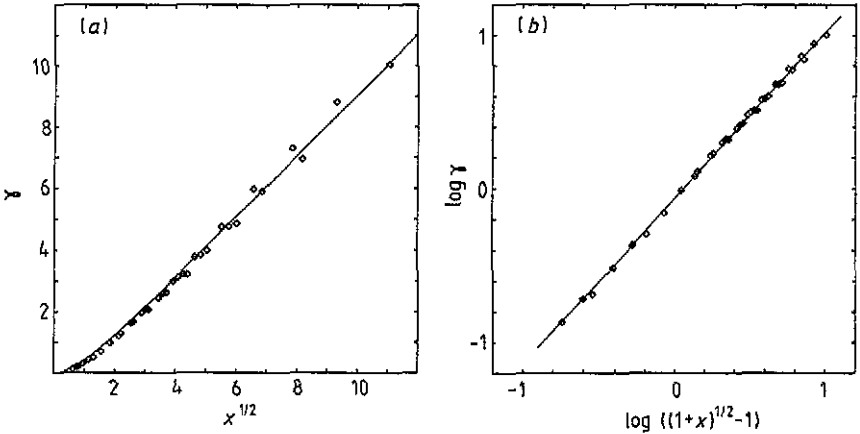

Figure 2. The parameter γ is plotted against $\sqrt{x}$ (a). It seems obvious that asymptotically the points lie on a straight line. The curve $\sqrt{1+x} - 1$ is drawn to show possible overall dependence. This assumption can be verified by looking at the plot of $\log \gamma$ against $\log(\sqrt{1+x} - 1)$ shown in (b). All points in (b) fall on a straight line $\log \gamma = c \log(\sqrt{1+x} - 1) + d$ with the parameter values $c = 1.08$ and $d = -0.06$.

δE can be calculated by estimating the scalar product of eigenvector y^j with the jth row of the matrix A_{kl} which should be equal to $E_j y_j^j$. One obtains $\delta E \approx \sqrt{m L_e / b}$ provided that the localization length L_g is of the same order or smaller than b, and that $m \gg 1$, so that $\delta E > 1$. The former condition is met for our parameter values but the latter is not since we assumed $m = 1$. Nevertheless we discovered that γ is a function of the scaling variable x only. The numerical data shown in figure 2 strongly support the hypothesis that in the case $m = 1$ we have

$$\gamma(\alpha, b, m = 1) = \alpha L_g \sqrt{\frac{b}{L_e}} = \sqrt{1+x} - 1 \approx \sqrt{x} \qquad \text{when } x \gg 1. \tag{21}$$

The small x functional dependence $\sqrt{1+x} - 1$ is perhaps only a good numerical approximation to $\gamma(x)$ but the asymptotic behaviour $x \gg 1$ is quite significantly given by $\gamma(x) \approx x^{1/2}$. It is interesting to note that in the case $m = b$ of BRME we must take $\delta E = \sqrt{b}$ and we have then $\gamma(x) = \alpha L_g / \delta E = \alpha L_g / \sqrt{b} = O(1) = \text{constant}$, as has been found analytically by Fyodorov and Mirlin (1991b), but has been wrongly predicted in (Wilkinson *et al* 1991) to behave as $\gamma(x) = O(x^{1/3})$. The latter behaviour *would* imply approach to GOE as $x \to \infty$, but in reality the Fyodorov–Mirlin result shows that this conclusion by Wilkinson *et al* (1991) is in error as seen above. We should emphasize that in our case $\gamma(x)$ does not control the level repulsion in the manner described before because the condition $m \gg 1$ is not fulfilled

since we have $m = 1$, e.g. large $x \gg 1$ implies large $\gamma \gg 1$ but small level repulsion exponent $\beta \ll 1$, i.e. no level repulsion.

Two final remarks are in order. Firstly, contrary to a naive belief the sparsity strongly affects the localization lengths and the scaling laws such as (21). This has been verified by looking numerically at the localization lengths L_g and L_e at small sparsity $s = m/b$ of the order of a few percent. At small x they are indeed typically of the order of b and *a fortiori* much smaller than $\approx 1.4b^2$, the latter being predicted for BRME by Wilkinson *et al* (1991), Casati *et al* (1990) and by Fyodorov and Merlin (1991b). Therefore, our scaling laws such as (21) *cannot* be directly compared with the predictions for BRME. Indeed, a naive substitution of $L_g \approx L_e = b^2 f(x)$ at large x where according to Fyodorov and Merlin (1991b) $f(x) \approx 1/x$, we find $\gamma(\alpha, b, m = b) \approx \sqrt{x/b}$, so that in contrast to the small sparsity case $m \ll b$ the quantity γ is no longer a function of x alone. Secondly, it should be emphasized that due to our clear physical motivation we study the SBRME at fixed value of m rather than at fixed sparsity $s = m/b$.

In any case, since in this work we are primarily interested in the approach of the level statistics towards the Poissonian behaviour the case $m = 1$ is most important and interesting, but also technically most easily studied.

We would like to mention that in order to obtain the numerical results of this paper it was necessary to use one month of CPU time on the VAX 8800 computer, which is therefore the maximum information on SBRME that we are presently able to extract. The careful study of the dependence of the scaling laws upon the parameter m must be therefore delegated to future work. For example, the preliminary study of the case $m = 2$ (using the same b's as before) shows that the quality of the Brody-fit deteriorates substantially, so that sampling from SBRME should be considerably larger in order to achieve the same statistical significance.

4. Conclusions

We have modelled some statistical properties (such as the spectral statistics and average localization lengths) of quantized Hamiltonian systems in the transition region between integrability and chaos (particularly KAM systems when the perturbation parameter is small) by means of newly defined sparsed banded random matrix ensemble (SBRME). The statistical assumptions in the definition of the ensemble were the neglecting of the correlations among the *non-zero offdiagonal matrix elements* and the assumption of the Gaussian randomness of their statistical distribution. The physical motivation for introducing SBRME stems from the fact exposed in detail in Prosen and Robnik (1992a) that Hamiltonian systems in the transition region between integrability and chaos only very slowly approach the semiclassical limit which is accurately described by the Berry–Robnik (Berry and Robnik 1984) formulae, but at small S and finite spectra they typically exhibit power law level repulsion, which is well described by the Brody-like distributions such as e.g. the Izrailev distribution (Izrailev 1989, Casati *et al* 1991). These statistical assumptions in the definition of SBRME (or only one of them) are obviously crucial for the semiclassical behaviour of the spectral fluctuations. We found that the spectral fluctuations of the small sparsity SBRME exhibit the power law level repulsion for $\epsilon \neq 0$ and tend to the Poissonian statistics as $\epsilon \to 0$, and thus behave quite universally, since the Brody formula (1) can be applied successfully. These results thus clearly confirm our expectations that

the average statistical properties of dynamical Hamiltonian systems such as the level repulsion are well captured by SBRME.

As for the other mathematical properties of the SBRME($\alpha, b, m = 1$) we have established two important numerical results: (i) the existence of the scaling law $\beta(x)$ of the level repulsion parameter β as a function of the scaling variable $x = \alpha b^{3/2}$ (α = the mean diagonal increment, b = bandwidth), and (ii) the existence of the scaling law $\gamma(x)$, $\gamma = \alpha L_g \sqrt{b/L_e}$, where L_e and L_g are the average entropy and the average geometric localization length (of the eigenvectors), respectively. Asymptotically we find $\gamma \approx \sqrt{x}$ with great statistical significance.

We believe that these results represent an interesting new direction of research towards understanding of the universal aspects of the statistical properties of quantized KAM Hamiltonians.

Acknowledgment

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